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Is Taylor's graph geometric?

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Abstract

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Taylor's graph is a strongly regular graph that is the unique descendant of a certain regular two-graph on q^3+1 points, where q is an odd prime power. It has the parameters of the point graph of a putative partial geometry $\mathrm{PG}(q-1, \frac{1}{2}(q^2-1), \frac{1}{2}(q-1))$ and so is pseudo-geometric. Here we investigate the question as to whether or not Taylor's graph is geometric and discover that it is when $q=3$ but not in the cases $q=5, 7$.

1. Introduction

We assume the reader is familiar with the basic facts about two-graphs, regular two-graphs and strongly regular graphs as can be found, for example, in the survey articles by Seidel [3, 4] and Seidel and Taylor [5].

In his paper [6] Taylor defines a class of regular two-graphs in the following way. Let H be a nondegenerate Hermitean form on the projective plane $\Pi = \mathrm{PG}(2, q^2)$, where q is an odd prime power, and let $\Omega = \{\boldsymbol{x} \in \Pi \mid H(\boldsymbol{x}, \boldsymbol{x}) = 0\}$. It is not difficult to see that $|\Omega| = q^3+1$. A regular two-graph can be constructed on Ω by taking the triples of the two-graph to be the triples $\{\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}\}$ such that

$$-H(\boldsymbol{x}, \boldsymbol{y})H(\boldsymbol{y}, \boldsymbol{z})H(\boldsymbol{z}, \boldsymbol{x}) \in B(q^2), \tag{1}$$

where $B(q^r)$ is the set of nonsquares in $\mathrm{GF}(q^r)$ if $q \equiv 1 \pmod 4$ and the set of squares if $q \equiv 3 \pmod 4$. Isolating a vertex ∞, say, by switching, and then deleting it gives a strongly regular graph with parameters

$$v = q^3, \qquad k = 2\mu = \tfrac{1}{2}(q-1)(q^2+1), \qquad \lambda = \tfrac{1}{4}(q-1)^3 - 1.$$

This is the graph we refer to as *Taylor's graph*. In Section 2 we examine this graph in more detail and derive some results about the structure of its cliques.

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A *partial geometry* $\mathrm{PG}(s, t, \alpha)$ is an incidence structure of points and lines such that every line has $s+1$ points, every point is on $t+1$ lines, two distinct lines meet in at most one point and for every non-incident point-line pair (p, l) there are α lines through p that meet l. It is not difficult to show that such a partial geometry has $(s+1)(st+\alpha)/\alpha$ points and $(t+1)(st+\alpha)/\alpha$ lines. The *point graph* of the partial geometry is the graph whose vertices are the points of the partial geometry, with vertices being adjacent if they are collinear. It is easily shown that this point graph is strongly regular with parameters $v=(s+1)(st+\alpha)/\alpha$, $k=s(t+1)$, $\lambda=t(\alpha-1)+s-1$, $\mu=\alpha(t+1)$. A strongly regular graph with these parameters is generally said to be *pseudo-geometric*, but if it is the point graph of some partial geometry it is called *geometric*. It is clear that Taylor's graph is pseudo-geometric for it has the parameters of the point graph of a putative $\mathrm{PG}(q-1, \frac{1}{2}(q^2-1), \frac{1}{2}(q-1))$. Is it in fact geometric? This is the question we examine in Section 3.

2. A characterisation of Taylor's strongly regular graph

For $x \in \mathrm{GF}(q^2)$ let $N(x)=x^{q+1}$ and $\mathrm{Tr}(x)=x+x^q$. We take the Hermitean form H mentioned in the introduction to be given by

$$H(\boldsymbol{x}, \boldsymbol{y})=x_1 y_1^q+x_2 y_3^q+x_3 y_2^q, \tag{2}$$

where $\boldsymbol{x}$ and $\boldsymbol{y}$ are given by $\boldsymbol{x}=(x_1, x_2, x_3)$ and $\boldsymbol{y}=(y_1, y_2, y_3)$, with x_i and $y_i \in \mathrm{GF}(q^2)$, $(1 \leqslant i \leqslant 3)$. Then $\boldsymbol{x} \in \Omega \Leftrightarrow N(x_1)+\mathrm{Tr}(x_2 x_3^q)=0$. Now the elements of Π are of the form (i) $(1,0,0)$, (ii) $(a, 1, 0)$, (iii) $(c, d, 1)$ and clearly the only element of type (ii) that is a vertex of Ω is $(0,1,0)$. Denote this vertex by ∞ so that Taylor's graph is defined on the vertex set $\Omega \backslash\{\infty\}$ by the rule

$$\{\boldsymbol{x}, \boldsymbol{y}\} \text{ is an edge } \Leftrightarrow \{\boldsymbol{x}, \boldsymbol{y}, \infty\} \text{ is a triple of the two-graph.}$$

Let $\boldsymbol{x}=(x_1, x_2, 1)$, and $\boldsymbol{y}=(y_1, y_2, 1)$, where $N(x_1)+\mathrm{Tr}(x_2)=N(y_1)+\mathrm{Tr}(y_2)=0$, be two vertices of Taylor's graph. By (1) and (2) they are adjacent if and only if

$$-(x_1 y_1^q+x_2+y_2^q) \in B(q^2). \tag{3}$$

We now write the elements of $\mathrm{GF}(q^2)$ in the form $a+ib$, $a, b \in \mathrm{GF}(q)$ where i^2 is a nonsquare of $\mathrm{GF}(q)$. Suppose then that $x_1=a_1+ia_2$, $x_2=c_1+ia_3$, $y_1=b_1=ib_2$, $y_2=d_1+ib_3$, so that

$$a_1^2-i^2 a_2^2=-2c_1, \qquad c_1^2-i^2 c_2^2=-2d_1. \tag{4}$$

On account of (4), the vertices $\boldsymbol{x}, \boldsymbol{y}$ can be identified with the triples (a_1, a_2, a_3) and (b_1, b_2, b_3), respectively and we can regard the vertex set $\Omega \backslash\{\infty\}$ as the set of elements of $\mathrm{GF}(q^3)$. Using the fact that, for $x \in \mathrm{GF}(q^2)$, $x \in B(q^2) \Leftrightarrow N(x) \in B(q)$, it is easily seen that, with this identification, the adjacency condition (3) becomes $(a_1, a_2, a_3) \sim (b_1, b_2, b_3) \Leftrightarrow$

$$[(a_1-b_1)^2-i^2(a_2-b_2)^2]^2-4i^2(b_1 a_2-b_2 a_1+a_3-b_3)^2 \in B(q). \tag{5}$$

Clearly, if a, b are fixed, $(a, b, x) \sim (a, b, y)$ $\forall x \neq y$ so that $C(a, b) = \{(a, b, x) \mid x \in \mathrm{GF}(q)\}$ is a clique of size q, which we call a *special clique*. Thus Taylor's graph has a partition into q^2 disjoint q-cliques achieving the Hoffman bound, (see for example [1, Section 2, Theorem 1]) so that every point outside such a clique has precisely $\frac{1}{2}(q-1)$ neighbours on it.

It is immediate that a condition equivalent to (5) is that $(a_1 - b_1, a_2 - b_2, b_1 a_2 - b_2 a_1 + a_3 - b_3)$ and $(0, 0, 0)$ are adjacent, and from this it is a straightforward matter to show that for fixed $a, b, c \in \mathrm{GF}(q)$ the map $(x, y, z) \to (x - a, y - b, ay - bx + z - c)$ is an automorphism of Taylor's graph. If, further, we regard the first two components of a triple representing a vertex as an element of $\mathrm{GF}(q^2)$ in the obvious way, then we have the following.

Proposition 1. *For fixed $\beta \in \mathrm{GF}(q^2)$, $\beta \neq 0$, the maps*
(i) $\phi_\beta^+ : (\xi, x) \to (\beta\xi, N(\beta)x)$,
(ii) $\phi_\beta^- : (\xi, x) \to (\beta\xi^q, -N(\beta)x)$,
are automorphisms of Taylor's graph.

As an immediate consequence of this we have the following.

Corollary 2. *Taylor's graph has a transitive group of automorphisms of order $2q^3(q^2-1)$. Furthermore, each vertex is stabilized by a subgroup of order $2(q^2-1)$.*

Proposition 3. *Through each point (ξ, x) not on $C(0, 0)$ there is a unique q-clique that meets $C(0, 0)$ in $\frac{1}{2}(q-1)$ points.*

Proof. Let $\beta \neq 1$ be a square in $\mathrm{GF}(q^2)$, say $\beta = \gamma^2$, such that $N(\gamma) = 1$. We shall first show that for all (ξ, x), not on $C(0, 0)$ $\phi_\beta^+(\xi, x) \sim (\xi, x)$. Write $\beta = a + ib$. By (5),

$$(\xi, x) \sim (\beta\xi, x) \Leftrightarrow N^2(\xi - \xi\beta) - 4i^2 N^2(\xi) b^2 \in B(q)$$

and since $N(\xi) \neq 0$ this is equivalent to $N^2(\beta - 1) - 4i^2 b^2 \in B(q)$, or, what is the same thing, $N(N(\beta - 1) - 2ib) \in B(q)$. But $N(\beta - 1) - 2ib = 2(1 - \beta)$ so we get the condition that $4N(\beta - 1) = 4 \cdot 2(1 - a) \in B(q)$. However, β, being the square of an element of norm 1 takes the form $\beta = x^2 + i^2 y^2 + i(2xy)$ where $x^2 - i^2 y^2 = 1$ and $y \neq 0$, so that $8(1 - a) = -16 i^2 y^2 \in B(q)$ and (ξ, x) and $\phi_\beta^+(\xi, x)$ are adjacent.

Now $\{\beta \in \mathrm{GF}(q^2) \mid \beta = \gamma^2, N(\gamma) = 1\}$ forms a cyclic group, G, say, of order $\frac{1}{2}(q+1)$ and consequently, $\{\phi_\beta^+(\xi, x) \mid \phi_\beta^+ \in G\}$ is a clique of size $\frac{1}{2}(q+1)$. Moreover, by Hoffman's bound (ξ, x) is adjacent to $\frac{1}{2}(q-1)$ vertices of $C(0, 0)$, each of which is fixed by G. It follows that these $\frac{1}{2}(q-1)$ vertices together with the $\frac{1}{2}(q+1)$ vertices in the orbit of (ξ, x) under the action of G form a clique of size q. By Hoffmans's bound it is the unique q-clique through (ξ, x) meeting $C(0, 0)$ in $\frac{1}{2}(q-1)$ points. $\square$

It follows from Proposition 3 that the $q^3 - q$ vertices of Taylor's graph not on $C(0, 0)$ can be partitioned into $2q(q-1)$ disjoint cliques of size $\frac{1}{2}(q+1)$, each of which meets a special clique in at most one point. Moreover, they all have a unique extension to a clique of size q meeting $C(0, 0)$ in $\frac{1}{2}(q-1)$ points. In the case $q > 3$ the transitivity of the graph shows that there are $2q^3(q-1)$ cliques of size q each of which meets a unique special clique in $\frac{1}{2}(q-1)$ points. Hence, counting the q^2 special cliques, we have the following.

Proposition 4. *If $q > 3$, Taylor's graph has at least $q^2(2q^2 - 2q + 1)$ cliques of size q.*

3. Conclusions

In order that Taylor's graph be geometric it is necessary and sufficient that there be a set of $\frac{1}{2}q^2(q^2+1)$ cliques of size q that pairwise meet in at most one point. Since no two q-cliques can have more than $\frac{1}{2}(q-1)$ points in common it is clear that when $q = 3$ the graph is in fact geometric, for in that case the lines of the partial geometry can be taken to be the 3-cliques. However, when $q > 3$ the picture is quite different. We have been unable to find a method that can be applied to all cases and have only results for $q = 5, 7$. We deal first with the case $q = 5$.

Proposition 5. *When $q = 5$ every nonspecial 5-clique meets some special 5-clique in 2 points.*

Proof. To see this, let C be a 5-clique that meets some special clique in one point only. By transitivity we may assume the special clique to be $C(0, 0)$ and the point to be $(0, 0, 0)$. Now $(0, 0, 0)$ has 48 neighbours not on $C(0, 0)$ and the maps ϕ_β^+, ϕ_β^- of Proposition 1 generate a group of order 48 that fixes $(0, 0, 0)$. Thus taking $i^2 = 2$, we may assume that $(1, 0, 1) \in C$. The remaining 3 points of C must come from the $\lambda = 15$ common neighbours of $(0, 0, 0)$ and $(1, 0, 1)$. Clearly then there can be at most 5 cliques of size 5 through these two points. In fact, detailed examination shows that there are only 3, namely

(i) $\{(0,0,0), (1,0,1), (0,0,2), (2,2,1), (2,3,1)\}$,
(ii) $\{(0,0,0), (1,0,1), (1,0,4), (4,2,3), (4,3,2)\}$,
(iii) $\{(0,0,0), (1,0,1), (3,1,2), (3,1,3), (3,3,4)\}$,

and each of these meets some special clique in 2 points. $\square$

As a consequence of this result, we deduce that the $q^2(2q^2 - 2q + 1) = 1025$ cliques of size 5 quoted in Proposition 4 comprise the complete set. We now introduce a partitioning of these cliques into 25 sets of 41 from which it will be easy to see that Taylor's graph is not geometric in this case.

Let C be one of the special cliques. For $a, b \in C$ let $\Gamma(a, b)$ denote the set of common neighbours of a, b not in C; then $|\Gamma(a, b)| = 12$. Also, different pairs (a, b) give rise to disjoint $\Gamma(a, b)$. Hence, as $\{a, b\}$ runs through all pairs of C, $\Gamma(a, b)$ covers all points not in C. Following from Proposition 3 it is clear that each $\Gamma(a, b)$ comprises 4 triangles so there are 4 other 5-cliques through a, b in addition to C. Since there are 10 such pairs $\{a, b\}$, we obtain a total of 40 cliques of size 5 that meet C in 2 points (and every other special clique in at most 1 point). Thus we have a partitioning of the 5-cliques into 25 sets, each set comprising a special clique and 40 others that can be split into ten sets of four, such that the four in the same set have the same two points of the special clique in common. In attempting to choose a set of 5-cliques that pairwise meet in at most one point we have at most 10 from each of the above 25 sets of 41, giving a total of at most 250. This falls short of the required number, namely 325, were Taylor's graph to be geometric.

When $q = 7$ we found by computer that there are 261 cliques of size 7 through the point $(0, 0, 0)$ (and hence through each point). These can be split into classes according to the number of points they have in common with $C(0, 0)$ and $H = \{(x, y, 0) \mid (x, y) \neq (0, 0)\}$, see Table 1.

For Taylor's graph with $q = 7$ to be the graph of a PG(6, 24, 3) it is necessary that every point be on 25 lines; in particular, there must be 25 cliques of size 7 having only the point $(0, 0, 0)$ in common. Let $a, b, \ldots, f$ denote the numbers from the below classes $A, B, \ldots, F$ that go to make such a set of 25, so that $a + b + c + d + e + f = 25$. Since this pencil of lines must cover all the neighbours of $(0, 0, 0)$ exactly once ($k = 150$ in total) we obtain

$$\begin{aligned} &6a + 2b + 2c = 6, \\ &4c + 4d + 2e + 6f = 48, \qquad (6) \\ &4b + 2d + 4e = 96. \end{aligned}$$

However, the graph obtained from B by taking the cliques as vertices, with two cliques being adjacent if they have only $(0, 0, 0)$ in common, turns out to be bipartite $K_{12,12}$, and hence has no cliques of size 3, so that $b \leq 2$. There are then

Table 1

Class	Number of cliques	Points of $C(0, 0)$	Points of H
A	1	7	0
B	24	3	0
C	12	3	4
D	48	1	4
E	168	1	2
F	8	1	6

only two solutions of (6):

a	b	c	d	e	f
1	0	0	0	24	0.
0	2	1	0	22	0

Thus any pencil of 25 lines through $(0, 0, 0)$ must contain at least 22 cliques from the class E and none from D, F.

Denote the lines of such a pencil by $l_1, l_2, \ldots, l_{25}$. Then for each i $(1 \leq i \leq 25)$ and each point $p \in l_i$, l_i will be one of the 25 lines of a pencil through p. As was pointed out in Section 2, for each $p = (a, b, c)$, say, Taylor's graph has an automorphism $\theta_p : (x, y, z) \to (x - a, y - b, ay - bx + z - c)$ and this induces an isomorphic copy of the partial geometry in which,

$$\text{for each } p \in l_i,\ \theta_p(l_i) \text{ is a line in a pencil of 25 through } (0, 0, 0). \tag{7}$$

Now the 168 cliques in class E can be partitioned into two subclasses, E' and E'', say, those in E' having 3 points in common with a unique special clique (48 in number) and the remainder, comprising 120 cliques, each of which has all its points belonging to different special cliques, being in E''. Investigation shows that each clique $l \in E''$ possesses exactly two points $p \neq (0, 0, 0)$, such that $\theta_p(l)$ contains four points of H and one of $C(0, 0)$, and so is in class D. Consequently, $\theta_p(l)$ cannot satisfy (7). It follows that any pencil of 25 lines through $(0, 0, 0)$ in the partial geometry must contain 22 cliques from E'. Define a graph G' on these 48 cliques with two cliques adjacent if they meet in more than one point. Then G' is regular of degree 6. Using MATLAB its smallest eigenvalue was found to be -3 and hence by the co-clique bound of Hoffman for regular graphs [2] the maximum co-clique size is 16 (This was also confirmed by a computer search which gave 9 co-cliques of size 16). Thus it is not possible to find 22 of these 48 cliques in E' meeting only in $(0, 0, 0)$. Hence Taylor's graph is not geometric when $q = 7$.

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